

Boole's conditions of possible experience and reasoning under uncertainty

Pierre Hansen^a, Brigitte Jaumard^{b,*}, Marcus Poggi de Aragão^c

^aGERAD, École des Hautes Études Commerciales, 5455 avenue Decelles, Montréal, Qué., Canada H3T 1V6

^bGERAD, École Polytechnique de Montréal, Département de Mathématiques Appliquées, Succursale A, Case Postale 6079, Montréal, Qué., Canada H3C 3A7

^cComputer Science Department, Universidade Estadual de Campinas, DCC/IMECC/UNICAMP, C.P. 6065 Campinas SP, Brazil 13081-970

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Abstract

Consider a set of logical sentences together with probabilities that they are true. These probabilities must satisfy certain conditions for this system to be consistent. It is shown that an analytical form of these conditions can be obtained by enumerating the extreme rays of a polyhedron. We also consider the cases when: (i) intervals of probabilities are given, instead of single values; and (ii) best lower and upper bounds on the probability of an additional logical sentence to be true are sought. Enumeration of vertices and extreme rays is used. Each vertex defines a linear expression and the maximum (minimum) of these defines a best possible lower (upper) bound on the probability of the additional logical sentence to be true. Each extreme ray leads to a constraint on the probabilities assigned to the initial set of logical sentences. Redundancy in these expressions is studied. Illustrations are provided in the domain of reasoning under uncertainty.

Keywords: Linear programming; Probabilistic satisfiability; Vertex and ray enumeration; Analytical solution

1. Introduction

Given a set of events and their probabilities, these probabilities must satisfy some conditions for this system to be consistent, i.e., to correspond to some possible experience. For instance, let A and B be two events, if $\text{prob}(A) = p$ and $\text{prob}(AB) = q$, then $q \leq p$ must hold. In Chapter XIX of his book of 1854 *An Investigation of The Laws of Thought* [1], as well as in several contemporary and subsequent publications [2–5], George Boole considers a general statement of the problem of determining

*Corresponding author.

these “conditions of possible experience” and proposes several algebraic ways to solve it approximately or exactly. A set of m logically defined events is given together with the probabilities that these events occur. As noted by Boole, it is equivalent to consider the probabilities to be true of propositions, asserting that these events occur. This amounts to describing the events by logical variables and the operators AND, OR and NOT. Boole next expresses each proposition as a sum of products, each product involving all logical variables in direct or complemented form. Unknown probabilities are then associated to each of these products. A set of equations is obtained expressing that, for each event, the sum of probabilities of the logical products for which this event occurs must be equal to its probability to occur. Elimination in these equations and in the nonnegativity constraints on the probabilities of variables corresponding to the probabilities of the products yields the conditions of possible experience. Moreover, Boole extends his method to solve the problem of finding the best possible lower and upper bounds on the probability of an additional event to occur. Boole [1] calls this last problem the “general problem” in the theory of probabilities.

More than a century later, Hailperin [10, 11] analyzes Boole’s methods and shows that the procedure described above is equivalent to Fourier [8] elimination, of which Boole was apparently unaware. In addition to clarifying various points about Boole’s conceptions (related in particular to the question of independence of events), Hailperin [10] makes two contributions. First, Hailperin shows that Boole’s general problem can be expressed as a linear program. This allows numerical solution of particular instances (in which the probabilities of the events are specified) by the simplex method. Further progress in this direction, using column generation techniques to solve large instances, has been made by Zemel [17], for a particular reliability problem, and by Georgakopoulos et al. [9], Kavvadias and Papadimitriou [13] and Jaumard et al. [12] in the general case. Second, Hailperin notes that an analytical expression of the lower and upper bounds of the probability of an event to occur can be obtained by enumeration of the vertices of the dual of the linear program expressing Boole’s general problem. To each vertex corresponds a linear expression in the probabilities of the events to occur. For given values of these probabilities, the lower (upper) bound is the largest (smallest) value for all such expressions. Hailperin’s model was rediscovered by Kounias and Marin [14] in their work on best linear Bonferroni bounds, and by Nilsson [16] in the context of artificial intelligence, under the name of probabilistic logic.

The purpose of the present paper is to complete Hailperin’s [10] analysis in the following way: first, it is shown that all conditions of possible experience can be obtained by enumerating the extreme rays of the polyhedron considered by Hailperin. It is next shown that there is no redundancy both in the linear expressions appearing in the bounds considered and in the conditions obtained. Analytical solution is then studied for an extension of Boole’s model, already suggested by Hailperin [10], in which probability intervals for the events to occur are given instead of single values.

A procedure is proposed to obtain all irredundant linear expressions in the lower and upper bounds and irredundant conditions of possible experience. In other words,

an algorithm is provided for determination of a complete analytical solution of the probabilistic satisfiability (PSAT) problem. Similar, not necessarily irredundant, expressions are given by a related procedure in the case where probability intervals are considered. A procedure to obtain irredundant expressions in this case too is sketched. Given such a solution, it suffices, for any set of numerical values of the probabilities of the events to occur, to substitute in the conditions to find whether the system is consistent (satisfiable) or not and in the expressions of the bounds to find best possible numerical values for them.

Finally, the strength of the analytical method is illustrated by automatic generation of complete analytical solutions for several sets of logical sentences with their probabilities of being true, arising when reasoning under uncertainty in expert systems.

2. Probabilistic satisfiability and linear programming

The probabilistic satisfiability problem [9,10,15] is defined as follows. Let $S = \{S_1, S_2, \dots, S_m\}$ be a set of m logical sentences defined on a set of n propositional variables $X = \{x_1, x_2, \dots, x_n\}$ and let $\pi = \{\pi_1, \pi_2, \dots, \pi_n\}$ be a set of probabilities that these sentences are true. Let $T = \{t_1, t_2, \dots\}$ denote the set of all possible assignments of the values true or false to the variables of X and $p = \{p_1, p_2, \dots\}$ denote a probability distribution on T . The question is then: does there exist a probability distribution p which satisfies the set of logical sentences together with their probabilities, such that for each sentence S_i ($i = 1, 2, \dots, m$) the sum of p_j 's over all truth assignments t_j which satisfy S_i equals π_i .

Let A be an $m \times |T|$ matrix such that a_{ij} is equal to 1 if the value assignment t_j satisfies S_i , and 0 otherwise. (Note that not all columns of A are necessarily distinct; the columns of A are called the *possible worlds* by Nilsson [16].)

The probabilistic satisfiability (PSAT) problem may then be stated: Is there a probability distribution p such that the system

$$\begin{cases} 1^t \cdot p = 1 \\ A \cdot p = \pi \\ p_j \geq 0 \quad \forall j = 1, \dots, |T| \end{cases}$$

has a solution?

We are now interested in necessary and sufficient conditions on the probability vector π which ensure a positive answer to the probabilistic satisfiability problem. The PSAT problem can be reformulated as follows:

Consider the linear program

$$(P) \quad \begin{cases} \min & 0 \cdot p \\ & 1^t \cdot p = 1 \\ & A \cdot p = \pi \\ & p_j \geq 0 \quad \forall j = 1, \dots, |T|. \end{cases}$$

Is there a feasible solution for (P)?

The dual of (P) is

$$(D) \quad \begin{cases} \max & y_0 + \pi \cdot y \\ & 1 \cdot y_0 + A^t y \leq 0 \end{cases}$$

and we have

Theorem 1. *The probabilistic satisfiability problem has a positive answer if and only if the inequality $(1, \pi)^t \cdot r \leq 0$ holds for all extreme rays r of (D).*

Proof. From the duality theorem of linear programming, the primal (P) is infeasible if the dual (D) is unbounded from above and is either infeasible or unbounded from below if the dual (D) is infeasible. Since $y_0 = 0$ and $y = 0$ is always a feasible solution for the dual (D), this dual (D) must be unbounded from above for (P) to be infeasible and conversely. Consider now a polyhedral description C of (D) by its extreme points and extreme rays:

$$C = \{x \in \mathbb{R}^n \mid x = \sum_{j=1}^l \lambda_j \cdot x^j + \sum_{j=1}^k \mu_j \cdot r_j; \sum_{j=1}^l \lambda_j = 1; \\ \lambda_j \geq 0, j = 1, 2, \dots, l; \mu_j \geq 0, j = 1, 2, \dots, k\},$$

where x^1, x^2, \dots, x^l are the extreme points and r_1, r_2, \dots, r_k the extreme rays of (D). (D) is unbounded from above if and only if there is at least one extreme ray r of (D) such that $(1, \pi)^t \cdot r > 0$. Consequently, $(1, \pi)^t \cdot r \leq 0$ must hold for all extreme rays r of (D) to obtain a positive answer to PSAT and this condition suffices. \square

Similar properties hold for linear programs in general (e.g. [15, p. 97]). They do not appear to have been applied to probabilistic satisfiability before. Considering an additional logical sentence S_{m+1} and seeking the best possible lower and upper bounds on its probability to be true leads to the optimization version of the PSAT problem (called “probabilistic entailment” by Nilsson [16]; in accordance with complexity theory we refer below to the probabilistic satisfiability problem for both its decision and its optimization versions). The corresponding linear programs have first been formulated by Hailperin [10] as an expression of Boole’s “general problem” in the theory of probabilities [1, p. 304]. They can be written as follows:

$$(Pmax) \quad \begin{cases} \max & \pi_{m+1} = A_{m+1} \cdot p \\ & 1^t \cdot p = 1 \\ & A \cdot p = \pi \\ & p_j \geq 0 \quad \forall j = 1, \dots, |T|. \end{cases}$$

$$(Pmin) \begin{cases} \min & \pi_{m+1} = A_{m+1} \cdot p \\ & \mathbb{1}^t \cdot p = 1 \\ & A \cdot p = \pi \\ & p_j \geq 0 \quad \forall j = 1, \dots, |T|. \end{cases}$$

and their respective duals as

$$(Dmin) \begin{cases} \min & y_0 + \pi \cdot y \\ & \mathbb{1} \cdot y_0 + A^t y \geq A_{m+1}^t \end{cases}$$

$$(Dmax) \begin{cases} \max & y_0 + \pi \cdot y \\ & \mathbb{1} \cdot y_0 + A^t y \geq A_{m+1}^t \end{cases}$$

where $A_{m+1} = (a_{m+1,j})$ is the $|T|$ -vector such that $a_{m+1,j}$ is equal to 1 if the value assignment t_j satisfies S_{m+1} , and 0 otherwise.

Hailperin [10] shows that the lower and upper bounds on π_{m+1} are given by a piecewise linear function of the probabilities π_i (for $i = 1, 2, \dots, m$), defined by the extreme points of the dual polyhedra $(Dmax)$ and $(Dmin)$, respectively. The following theorem summarizes this result. For completeness, we give a short proof of it.

Theorem 2 (Hailperin [10]). *The best lower (upper) bound for π_{m+1} is given by the following convex (concave) piecewise linear function of the probability assignment:*

$$\pi_{m+1}(\pi) = \max_{j=1,2,\dots,k_{\max}} (1, \pi)^t \cdot y_{\max}^j \quad \left(= \min_{j=1,2,\dots,k_{\min}} (1, \pi)^t \cdot y_{\min}^j \right),$$

where y_{\max}^j (y_{\min}^j) for all j represent the k_{\max} (k_{\min}) extreme points of $(Dmax)$ ($(Dmin)$).

Proof. For a fixed probability assignment π , the best bound is the optimum of a standard linear program $((Dmin)$ or $(Dmax)$). Consequently it will arise at one, or in case of dual degeneracy in the final tableau at several, of its extreme points. These extreme points are independent of π . Consider now all possible probability assignments π . The value of $y_0 + \pi y$ at any extreme point y_{\min}^j or y_{\max}^j is a linear function of π . The best bound is the maximum (minimum) of these linear functions. It is therefore a convex (concave) piecewise linear function of π (e.g., [15, p. 42]). \square

So finding general expressions for best possible lower and upper bounds on π_{m+1} reduces to vertex enumeration, on $(Dmax)$ or $(Dmin)$. As seen above, the conditions of possible experience are obtained by enumeration of the extreme rays of (D) . These conditions need not be obtained separately as shown by the next result.

Proposition 1. *The extreme rays of (D) coincide with those of $(Dmax)$ and are symmetric to those of $(Dmin)$.*

Proof. This follows directly from the fact that the polyhedra D and D_{\max} have the same cones of feasible directions: $C_D = C_{D_{\max}} = \{(y_0, y) | 1^t y_0 + A^t y \leq 0\}$. Moreover, the cones of feasible directions $C_{D_{\max}}$ and $C_{D_{\min}}$ of (P_{\max}) and (P_{\min}) are symmetric. \square

Hailperin's result can also be extended to characterize conditions of possible experience for (P_{\max}) and (P_{\min}) .

Corollary 1. *The probabilistic satisfiability problem has a complete analytical solution:*

$$\pi_{m+1} \geq \max_{j=1, 2, \dots, k_{\max}} (1, \pi)^t \cdot y_{\max}^j \left(\leq \min_{j=1, 2, \dots, k_{\min}} (1, \pi)^t \cdot y_{\min}^j \right)$$

subject to

$$(1, \pi)^t \cdot r \leq 0 \quad \text{for all extreme rays } r \text{ of } (D).$$

It follows from Corollary 1 that the probabilistic satisfiability problem reduces to vertex and extreme ray enumeration for polyhedra. Methods for vertex enumeration often rely on search of the adjacency graph of the given polyhedron (whose vertices and edges correspond to those of this polyhedron); they can easily be extended in order to enumerate extreme rays as well. A recent survey and computational comparison of methods for vertex enumeration is given in [7]. Such methods are applied to a few examples in Section 4.

3. Redundancy analysis

In this section we study whether there is some redundancy in the analytical expressions for the probability bounds and conditions of possible experience obtained as discussed in the previous section. We first examine whether each extreme point of the polyhedron $P_{D_{\max}}$ of (D_{\max}) (or similarly $P_{D_{\min}}$ of (D_{\min})) corresponds to the optimal solution of (D_{\max}) for some feasible probability assignment (possible experience). It turns out to be the case.

Theorem 3. *Consider the description of the polyhedron $P_{D_{\max}}$ of (D_{\max}) , by its extreme points x^1, x^2, \dots, x^l and extreme rays r_1, r_2, \dots, r_k . For all x^j ($j = 1, 2, \dots, l$), there exists a vector $(1, \pi)^t$ satisfying*

- (i) $\pi_i \in [0, 1]$ for $i = 1, 2, \dots, m$;
- (ii) $(1, \pi)^t \cdot r_i \leq 0$, $i = 1, 2, \dots, k$

and such that

$$(1, \pi)^t \cdot x^j = \max_{x \in P_{D_{\max}}} (1, \pi)^t \cdot x,$$

i.e., there is a probability assignment for which x^j is optimal.

Proof. Condition (ii) expresses only that (D_{\max}) is bounded and thus does not eliminate any extreme point of $P_{D_{\max}}$ from the list of potential optimal solutions. Proving no linear expression $(1, \pi)^T \cdot x^j$ is redundant amounts to showing that the direction cone defined by condition (ii), i.e.,

$$C_D = \{d \mid d^T \cdot r_i \leq 0, i = 1, 2, \dots, k\},$$

is contained in the cone defined by condition (i), i.e.,

$$C_{(i)} = \{d \mid d = \alpha \cdot v, \text{ where } v \text{ is such that } v_0 = 1$$

$$\text{and } v_j \in [0, 1] \forall j = 1, 2, \dots, m; \alpha \geq 0\}.$$

In other words, one must show that if (D_{\max}) has a bounded optimal value then all components of π belong to $[0, 1]$. To see this, consider the cone defined by the positive combinations of the columns of (P_{\min}) (the dual of (D_{\max})),

$$C_P = \left\{d \mid d = \sum_{j=1}^{|T|} \begin{pmatrix} 1 \\ a^j \end{pmatrix} p_j; p_j \geq 0, j = 1, 2, \dots, |T|\right\},$$

where a^j denotes the j th column of A , and note that (P_{\min}) is feasible if and only if its right-hand side belongs to C_P . Since C_P and C_D are two descriptions of the set of feasible π vectors for (P_{\min}) they are equal. Consider the following representation of cone $C_{(i)}$ by generating directions:

$$C_G: \left\{d \mid d = \sum_{j=0}^m \lambda_j \cdot e_j; \lambda_j \leq \lambda_0; \lambda_j \geq 0, j = 1, 2, \dots, m \text{ and } \lambda_0 > 0\right\},$$

where e_j is the unit vector with a one for the $(j + 1)$ th component and zeros elsewhere. Then, it is easy to verify that any generating direction of the cone C_P can be obtained as a positive combination of the generating directions of C_G . The result follows. \square

Proving non-redundancy of the constraints generated by the extreme rays is straightforward. First, recall that an extreme ray cannot be expressed as a linear combination of the other ones. Therefore, it defines a facet of the feasibility cone. Second, as shown above, the cone C_D described by the extreme rays is contained in or equal to the cone defined by the component constraints $C_{(i)}$. Therefore, no facet of the polyhedral cone C_D , defined by an extreme ray of P_D , is redundant, i.e., none is strictly outside $C_{(i)}$.

4. Interval probabilistic satisfiability

Hailperin [10] proposes an extension of probabilistic satisfiability in which probability intervals $[\pi_i^{\text{int}}, \pi_i^{\text{up}}]$ are assigned to the logical sentences S_i instead of the single probability values π_i for $i = 1, 2, \dots, m$. The model so obtained is often more realistic in applications of reasoning under uncertainty than the previous one. Hailperin [10] shows that Fourier elimination and linear programming methods can be readily extended to obtain analytical and numerical best possible bounds. The same is true for

column generation techniques as shown in [12]. Nilsson [16] also briefly discusses the use of probability intervals in probabilistic satisfiability. He suggests solving two linear programs in which the probabilities of the logical sentences are set to π_i^{inf} in the first case and to π_i^{sup} in the second one. As shown below, this may lead to incorrect probability bounds.

We now discuss how to get a complete analytical solution for interval probabilistic satisfiability. The primal probabilistic satisfiability problem can be written

$$(P_{m+1-\text{int}}) \begin{cases} \min/\max & A_{m+1} \cdot p \\ & \mathbf{1}^t \cdot p = 1 \\ & \pi_{\text{inf}} \leq A \cdot p \leq \pi_{\text{sup}} \\ & p_j \geq 0 \quad \forall j = 1, \dots, |T|. \end{cases}$$

Straightforward extensions of Theorems 1 and 2 show that we need to obtain the description by extreme points and extreme rays of the polyhedra defined by the feasible solution sets of

$$(D_{\text{max_int}}) \begin{cases} \max & y_0 + \pi^{\text{sup}} \cdot y + \pi^{\text{inf}} \cdot y' \\ & \mathbf{1} \cdot y_0 + A^t \cdot y + A'^t \cdot y' \leq A_{m+1}^t \\ & y \leq 0, \quad y' \geq 0. \end{cases}$$

$$(D_{\text{min_int}}) \begin{cases} \min & y_0 + \pi^{\text{sup}} \cdot y + \pi^{\text{inf}} \cdot y' \\ & \mathbf{1} \cdot y_0 + A^t \cdot y + A'^t \cdot y' \geq A_{m+1}^t \\ & y \geq 0, \quad y' \leq 0. \end{cases}$$

Again standard algorithms for extreme points and extreme rays enumeration can be applied. A comparison of complete analytical solutions for a small set of sentences in the single probability value and in the probability interval cases is given in Example 1 of Tables 1 and 2. Each constraint is generated by an extreme ray of $(D_{\text{max_int}})$ (or $(D_{\text{min_int}})$) and each expression for the lower (upper) bound is given by an extreme point of $(D_{\text{max_int}})$ ($(D_{\text{min_int}})$) with no extreme point or ray being omitted from this table (and from the following ones). Using probability intervals instead of single values clearly leads to a large increase in the number of constraints and linear expressions in the bounds. Before comparing the bounds obtained in both cases, we discuss Nilsson's proposal, i.e., to substitute all π_i by π_i^{inf} and then by π_i^{sup} . Although Nilsson does not clarify when to maximize or minimize, the following example shows that in all possible cases his suggestion eventually leads to incorrect probability bounds.

Example 4.1. Consider the set of logical sentences in Example 1 of Table 2 together with the probability intervals: $\pi_1 \in [0.3, 0.4]$, $\pi_2 \in [0.4, 0.4]$, $\pi_3 \in [0.2, 0.3]$, $\pi_4 \in [0.4, 0.4]$ and $\pi_5 \in [0.5, 0.5]$.

Solving the linear program for maximization or minimization with the probabilities all set at their upper bound or all at their lower bound, leads in both cases

Table 1
Complete analytical solutions for logical systems with single probability values for the truth of sentences

Example	Rules	Probability assigned	Conditions of possible experience	Lower bound on π ? Maximum of:	Upper bound on π ? Minimum of:
1	X_1 X_2 $X_1 \wedge X_3$ $X_2 \wedge X_3$ $\bar{X}_1 \wedge \bar{X}_2 \wedge X_3$ X_3	π_1 π_2 π_3 π_4 π_5 $\pi?$	$\pi_3 \geq 0, \pi_4 \geq 0, \pi_5 \geq 0$ $\pi_1 \geq \pi_3$ $\pi_2 \geq \pi_4$ $\pi_1 + \pi_5 \leq 1$ $\pi_2 + \pi_5 \leq 1$ $\pi_3 + 1 \geq \pi_1 + \pi_4 + \pi_5$ $\pi_4 + 1 \geq \pi_2 + \pi_3 + \pi_5$	$\pi_3 + \pi_5$ $\pi_4 + \pi_5$	$(1 - \pi_1) + \pi_3$ $(1 - \pi_2) + \pi_4$ $\pi_3 + \pi_4 + \pi_5$
2a	X_1 $X_1 \rightarrow X_2$ X_2	π_1 π_2 $\pi?$	$\pi_1 + \pi_2 \geq 1$ $\pi_i \leq 1, i = 1, 2$	$\pi_1 + \pi_2 - 1$	π_2
2b	X_1 $X_1 \rightarrow X_2$ $X_2 \rightarrow X_3$ X_3	π_1 π_2 π_3 $\pi?$	$\pi_1 + \pi_2 \geq 1$ $\pi_2 + \pi_3 \geq 1$ $\pi_i \leq 1, i = 1, 2, 3$	$\pi_1 + \pi_2 + \pi_3 - 2$ 0	π_3
3	X_1 X_2 $X_1 \vee X_2 \rightarrow X_3$ X_3	π_1 π_2 π_3 $\pi?$	$\pi_1 + \pi_2 + \pi_3 \geq 1$ $\pi_i \leq 1, i = 1, 2, 3$	$\pi_1 + \pi_3 - 1$ $\pi_2 + \pi_3 - 1$ 0	π_3
4	X_1 X_2 $X_1 \wedge X_2 \rightarrow X_3$ X_3	π_1 π_2 π_3 $\pi?$	$\pi_1 + \pi_3 \geq 1$ $\pi_2 + \pi_3 \geq 1$ $\pi_i \leq 1, i = 1, 2, 3$	$\pi_1 + \pi_2 + \pi_3 - 2$ 0	π_3
5	X_1 X_2 $X_1 \rightarrow X_3$ $X_2 \rightarrow X_3$ X_3	π_1 π_2 π_3 π_4 $\pi?$	$\pi_3 + 1 \geq \pi_2 + \pi_4 \geq 1$ $\pi_4 + 1 \geq \pi_1 + \pi_3 \geq 1$ $\pi_i \leq 1, i = 1, 2, 3, 4$	$\pi_1 + \pi_3 - 1$ $\pi_2 + \pi_4 - 1$	π_3 π_4

to an optimal solution of value 0.9 (i.e., $\pi_{m+1} \in [0.9, 0.9]$). However, solving the linear program with the probability intervals gives $\pi_{m+1} \in [0.9, 1.0]$.

The expressions obtained for the bounds by the enumeration of the extreme points for (*Dmax_int*) (or for (*Dmin_int*)), i.e., for the probability interval case, are not always irredundant. This can again be seen clearly in Example 1 of Table 2: the lower bound for the interval case has four extreme points that are redundant (i.e., those corresponding to the expressions 0, π_3^{inf} , π_4^{inf} and π_5^{inf}). The reason for this is that from the primal problem all that can be said referring to the probability intervals assigned is that $\pi_i^{\text{inf}} \leq 1$, $\pi_i^{\text{sup}} \geq 0$ and $\pi_i^{\text{inf}} \leq \pi_i^{\text{sup}}$. So the cone defined by the boundedness of the value of the optimal dual solution is not contained in (and, in fact, contains) the cone defined by the implicit probability constraints, which are: $0 \leq \pi_i^{\text{inf}} \leq \pi_i^{\text{sup}} \leq 1$.

The remark above also stands for the constraints obtained for the extremes of the probability intervals assigned to the logical sentences. This means that the extreme rays define constraints that can be eliminated when the implicit constraints $0 \leq \pi_i^{\text{inf}}$ and $\pi_i^{\text{sup}} \leq 1$ are considered. This is the case for the last four constraints of the probability interval case in Example 5 of Table 2.

Nevertheless, the procedure used for vertices and rays enumeration can be modified to generate only the irredundant linear expressions and constraints. At any vertex, the directions going towards all neighboring vertices or defining extreme rays are available from the current tableau. Testing whether a vertex corresponds to an irredundant expression amounts to verifying that all these directions have a negative internal product with at least one vector in the cone of feasible directions (note that this cone, as all cones considered in this paper, is pointed). This follows from the fact that all directions for which a vertex is an optimal solution have a negative internal product with all directions of edges departing from this same vertex. Moreover, in the case studied here, a necessary condition for a direction to have a positive internal product with another one in the cone of feasible directions is that it has at least one positive component. Furthermore, as all other constraints required for a direction to be feasible are already imposed by the problem structure (which is shown above), this condition suffices. The verification is, then, an easy task.

The procedure to generate irredundant vertices and rays would thus consist of starting at any vertex, choosing a feasible direction and iterating until the optimal vertex for this direction is found. Next, a standard depth first search is done through all vertices with the additional condition of never entering an edge with a direction which has a symmetric with no positive component, i.e., that has no negative component.

5. Uncertainty in classical inference rules

Results of the previous sections are next applied to reasoning under uncertainty. To this effect we consider several sets of logical sentences which correspond to events and classical inference rules. Instead of assuming events to be certain or impossible and inference rules always to be correct, probabilities or probability intervals (expressing beliefs) that they occur or are valid are assigned to them.

For instance, in classical logic, the *modus ponens* inference rule says that if event A is verified and rule $A \rightarrow B$ is valid, then we certainly know that event B will be verified. Our concern is to determine the probability of truth for the occurrence of event B when all which is known about event A and rule $A \rightarrow B$ are probabilities (that A is verified and $A \rightarrow B$ is valid). Probability intervals for the truth of conclusions and consistency conditions are presented in Tables 1 and 2 for several inference systems.

Results in these tables can be viewed as automatically generated theorems. For instance, Example 4 of Table 1 can be written:

If events x_1 and x_2 have probability π_1 and π_2 and the inference rule " $(x_1 \text{ and } x_2) \text{ implies } x_3$ " has a probability π_3 then $\pi_1 + \pi_3 \geq 1$ and $\pi_2 + \pi_3 \geq 1$ must hold and the

probability for x_3 to occur is between $\max\{\pi_1 + \pi_2 + \pi_3 - 2, 0\}$ and π_3 . Moreover, these bounds are the best possible.

It is interesting to compare system 3 and system 5 of Table 1. The reason is that most expert systems (e.g., the Mycin system [6]) do not deal with the case where a disjunction of propositional variables is found in the implicant of a rule. Instead, they divide the disjunction to obtain several single implications. By comparing 3 with 5, setting $\pi_3 = \pi_4$ in 5, we obtain the same lower and upper bounds for the truth value of x_3 . The difference lies only in the consistency conditions, which are stronger for system 5.

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